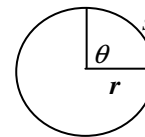


Trigonometry Review

Radian Measure

Radian measure of an angle is the ratio of the arc length s to the radius r .



In this example, since θ is a right angle, the arc length s is one-fourth of the

circumference, ie. $s = \frac{1}{4} 2\pi r = \frac{\pi r}{2}$ so that $\theta = \frac{s}{r} = \frac{\pi}{2}$. We see that $\frac{\pi}{2} = 90^\circ$. Now, since $\frac{\pi}{2} = 90^\circ$, it easily follows that $\pi = 180^\circ$, $2\pi = 360^\circ$, $\frac{\pi}{6} = 30^\circ$, $\frac{\pi}{4} = 45^\circ$ and $\frac{\pi}{3} = 60^\circ$.

Degree \leftrightarrow Radian

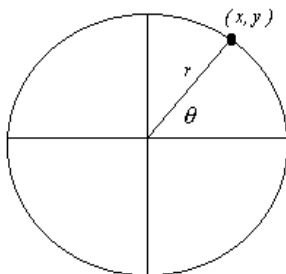
To convert from Degrees to Radians multiply by $\pi/180^\circ$.

To convert from Radians to Degrees multiply by $180^\circ/\pi$.

Sine and Cosine Functions

When an angle is given, it can always be drawn in terms of a circle with some radius r . Note that the radius of the circle is

$$r^2 = x^2 + y^2.$$



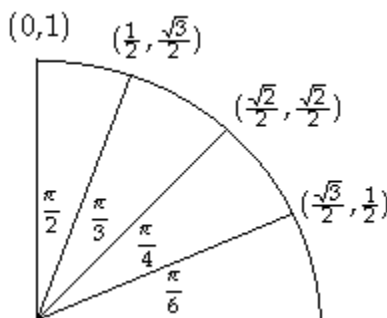
Trig functions are just standardized relationships between the values of x , y , r , and θ . The sine and cosine functions are defined as:

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r}.$$

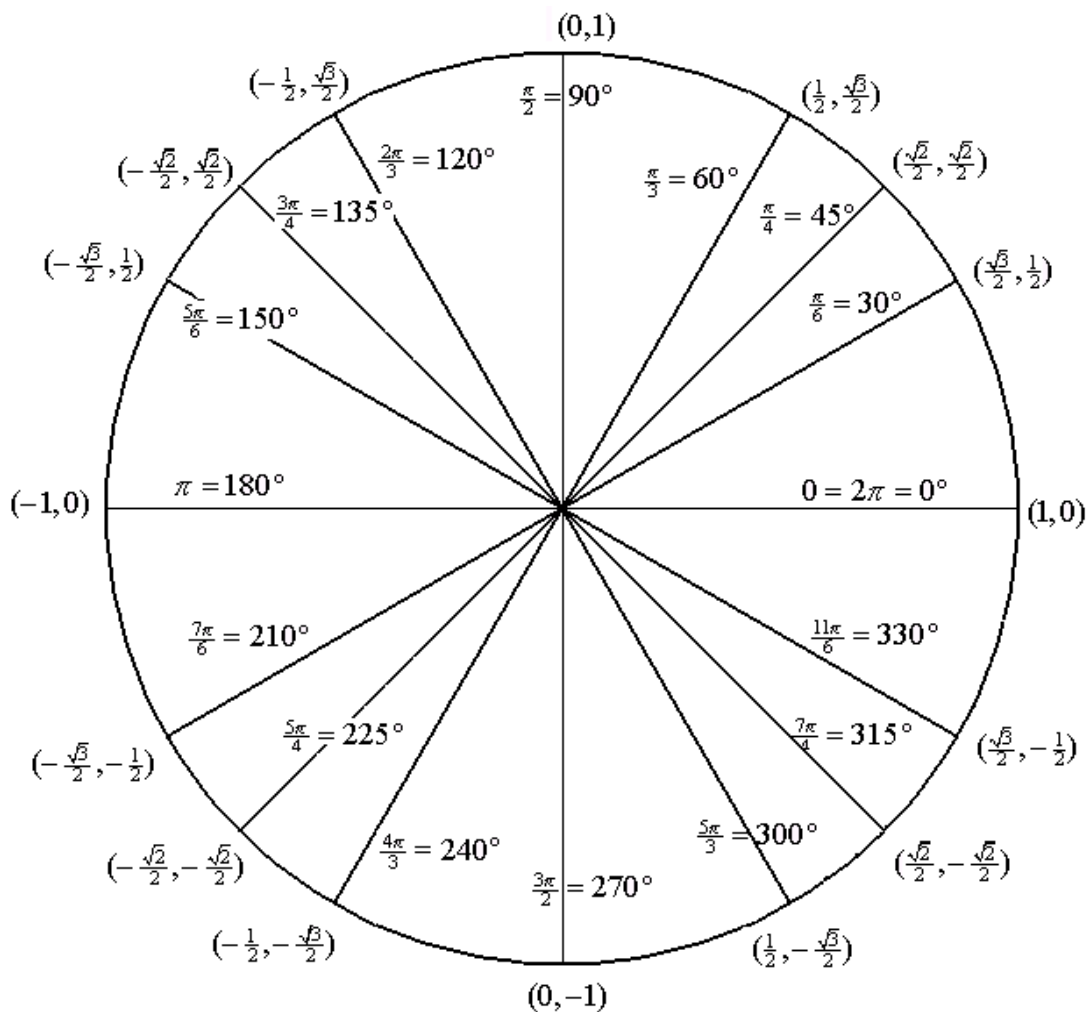
For simplicity, we often use the *Unit Circle* to remember the values of sine and cosine for special angles. In this case, $r = 1$ so $\sin \theta = y$ and $\cos \theta = x$ for the unit circle.

The Unit Circle

Special angles are those that are multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ from 0 to 2π . The values of sine and cosine for the special angles in the first quadrant can be found in the following figure:



The values of sine and cosine can be found for all other special angles by symmetry. Note that, except for the sign, the values of the trig functions at any given angle are the same as the values of the trig function at the reference angle (that is, the angle formed by going from the line to the x -axis). For example, the values of the trig function – with the exception of the sign – of $\frac{7\pi}{6}$ are the same as those for $\frac{\pi}{6}$.



For example, you can see that the values of sine and cosine are the following for selected angles:

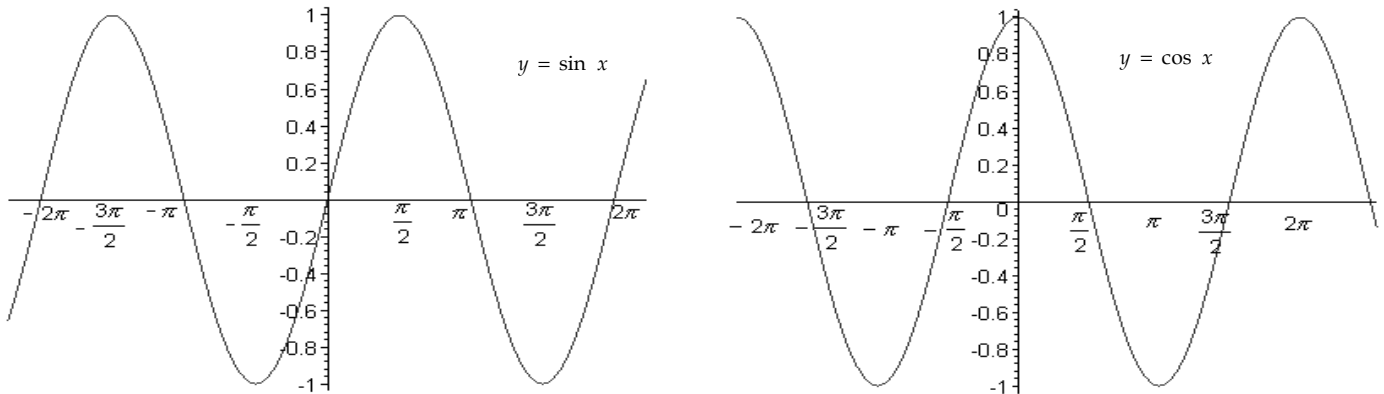
θ	$\sin \theta$	$\cos \theta$
0	0	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1	0
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
π	0	-1
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	-1	0
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
2π	0	1

Graphs of Sine and Cosine

Note the following:

As x increases from	$y = \sin x$	$y = \cos x$
0 to $\frac{\pi}{2}$	Positive Increases from 0 to 1	Positive Decreases from 1 to 0
$\frac{\pi}{2}$ to π	Positive Decreases from 1 to 0	Negative Decreases from 0 to -1
π to $\frac{3\pi}{2}$	Negative Decreases from 0 to -1	Negative Increases from -1 to 0
$\frac{3\pi}{2}$ to 2π	Negative Increases from -1 to 0	Positive Increases from 0 to 1

This leads to the following graphs for sine and cosine:



Other Trig Functions

All other trig functions are defined in terms of sine and cosine:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

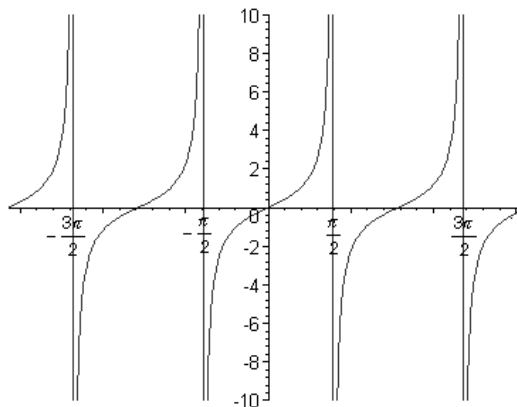
$$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

The Graph of the Tangent Function

Since $\tan x = \frac{\sin x}{\cos x}$, the tangent function will be undefined (i.e. has a vertical asymptote) wherever $\cos x$ is zero. Also, $\tan x$ will be zero wherever $\sin x$ is zero. By plotting a few points, we arrive at the following graph:



Basic Identities

Identities refer to the relationship between various trig functions. Here are a few:

$$\sin(-x) = -\sin x \quad \leftarrow \text{Note that this means that sine is an odd function}$$

$$\cos(-x) = \cos x \quad \leftarrow \text{Note that this means that cosine is an even function}$$

$$\tan(-x) = -\tan x \quad \leftarrow \text{Tangent is an odd function}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2(\sin x)(\cos x)$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Trigonometric Equation

Some trigonometric equations are easy. For example:

$$\text{Solve: } 2 \cos \theta = 1.$$

Solution: We can easily transform the equation into $\cos \theta = \frac{1}{2}$. Now to find all angles θ that satisfy this, look at the unit circle. We get $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$. Note, however, that these are just the solutions from 0 to 2π . To get *all* solutions, we just go around the circle again (and again, and again, and again, and again...). This just means that we add multiples of 2π . So the solutions are $\theta = \frac{\pi}{3} + 2\pi k$ and $\theta = \frac{5\pi}{3} + 2\pi k$ where we assume that k is an integer.

It gets a little trickier when there is a multiple of an angle in the equation.

$$\text{Solve: } \sin(4\theta) = 0.$$

Solution: What we do is we begin by turning it into an equation that doesn't have a multiple angle in it by letting some variable equal 4θ , say $Z = 4\theta$. Now we have the equation $\sin(Z) = 0$ which is easily solved by looking at the unit circle. We get our answers to be $Z = 0, \pi$, and 2π . Here is the tricky part, these are the solutions for Z from 0 to 2π but that doesn't mean that we have found all solutions for θ between 0 and 2π . You can see that if θ goes from 0 to 2π , this means that Z must go from 0 to 8π (since $Z = 4\theta$). In other words, to ensure that we get all solutions for θ from 0 to 2π , we need to first find all solutions for Z from 0 to 8π . These solutions are $Z = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi$ and 8π . Now, since $Z = 4\theta$, we have $4\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi$ which leads to the solutions $\theta = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4}, \frac{8\pi}{4}$ or $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$. These are all the solutions for θ from 0 to 2π . To get *all* solutions, we add multiples of 2π . We can write the answer as $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi + 2\pi k$.